ca implies b = c for all a, b, c in G then G is Abelian (that is, left-right cancellation property implies Abelian). (2+2.5+3)

(c) Show that the set $G = \{\begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0\}$

is a group under matrix multiplication. (7.5)

6. (a) State two-step subgroup test. Let G be an Abelian group and H, K be subgroups of G then show that

$$HK = \{hk \mid h \in H, k \in K\}$$

is a subgroup of G. (2+5.5)

(b) Define order of an element 'a', $O(a)$,, in a group G. Prove that in any group G, $O(bab^{-1}) = O(a)$ for all $a, b \in G$.. (2+5.5)

(c) Write all the generators of the cyclic group Z_{24}. Further describe all the subgroups of Z_{24} and find all generators of the subgroup of order 8 in Z_{24}. (3+3+1.5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1363 I

Unique Paper Code : 2352011101

Name of the Paper : Algebra (DSC-1)

Name of the Course : B.Sc. (H) Mathematics

Semester : I

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.
2. Attempt all questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.

1. (a) (i) Find a cubic equation with real coefficients two of whose roots are 1 and 3+2i. Also state the result being used.

(ii) Find an upper limit (using both the theorems) to the roots of

$$x^7 + 2x^5 + 4x^4 - 8x^2 - 32 = 0. \quad (3.5+4)$$

(b) Solve $3x^3 + 11x^2 + 12x + 4 = 0$, being given that roots are in Harmonic progression. (7.5)

(c) Find all the rational roots of $6y^3 - 11y^2 + 6y - 1 = 0$. (7.5)

2. (a) Compute $z^n + \frac{1}{z^n}$ if $z + \frac{1}{z} = \sqrt{3}$. (7.5)

(b) Find |z|, arg z, Arg z, arg(-z) for $z = (7-7\sqrt{3}\ i)(-1 -i)$. (7.5)

(c) Solve the equation $z^7 - 2iz^4 - iz^3\ 2 = 0$. (7.5)

3. (a) Solve $28x^3 + 9x^2 - 1 = 0$ by Cardan's method. (7.5)

(b) If a, b and c are non-zero integers with a and c relatively prime, prove that gcd (a, bc) = gcd (a, b) (7.5)

(c) (i) Find gcd of 1800 and 756 and express it in the form ma + nb for some integers m and n.

(ii)If a and b are relatively prime integers, prove that gcd (a + b, a - b) = 1 or 2. (4+3.5)

4. (a) Solve the following pair of congruences, if possible. If no solution exists, explain why not.

$$2x + y \equiv 1 \pmod 6$$

$$x + 3y \equiv 3 \pmod 6$$

(7.5)

(b) If $a \equiv x \pmod n$ and $b \equiv y \pmod n$, then

(i) $a + b \equiv x + y \pmod n$

(ii) $ab \equiv xy \pmod n$ (3.5+4)

(c) State fundamental theorem of arithmetic. Suppose a and b are integers and p is a prime such that plab. Prove that $p|a$ or $p|b$. (2.5+5)

5. (a) Describe symmetries of a non-square rectangle with diagrams. Also, construct the corresponding Cayley table. (3.5+4)

(b) Define an Abelian group. Show that in a group G if ab = ac then b = c (called left cancellation property). Further, show that in a group G if ab =

(d) State D'Alembert's Ratio test for a series. Find if the series,

$$\frac{1}{2}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+\cdots$$ is convergent.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1382 **I**

Unique Paper Code : 2352011102

Name of the Paper : Elementary Real Analysis

Name of the Course : **B.Sc. (H) Mathematics (NEP-UGCF 2022)**

Semester : I – DSC-2

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. Part of the questions to be attempted together.
4. **All** questions carry equal marks.
5. Use of Calculator is not allowed.

1. (a) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

(b) State the order properties of $\mathbb{R}$. Using it prove that if a, b, c are real numbers such that $a > b$, then $a + c > b + c$.

(c) Find all values of x satisfying $|x - 2| \leq x + 1$.

(d) Write the definition of Supremum and Infimum of a set. Give an example of a set having supremum and infimum, where the set

(i) contains its supremum and infimum

(ii) does not contain its supremum and infimum

2. (a) State and prove Archimedean property.

(b) Let 5 be a non-empty subset of $\mathbb{R}$ and $a > 0$, then show that

$$\sup(aS) = a \sup S$$

6. (a) State the Alternating Series test. Show that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is convergent.

(b) Test the convergence of the series

$$\frac{1}{e} + \frac{4}{e^2} + \frac{27}{e^3} + \frac{256}{e^4} + \frac{3{,}125}{e^5} + \cdots.$$

(c) Define a conditionally convergent series and an absolutely convergent series. Test the series

$\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^{3/2}}$ for absolute or conditional convergence.

5. (a) State and prove Cauchy Criterion for convergence of a series $\sum_{n=1}^{\infty} a_n$.

(b) Test the convergence of the following series :

(i) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

(c) Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, $p > 0$ is convergent for $p > 1$ and divergent for $p \leq 1$.

(d) Show that if the series Σu_n converges, then $\lim_{n\to\infty} u_n = 0$. Is the converse true? Justify your answer.

(c) Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim_{n\to\infty}(a_n) = 0$ and for some constant $K > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq K a_n$ for all $n \geq m$, then prove that $\lim_{n\to\infty}(x_n) = x$.

(d) Using the definition of limit, show that

$$\lim_{n\to\infty}\left(\frac{4n+5}{3n+4}\right) = \frac{4}{3}.$$

3. (a) Let (x_n) and (y_n) be sequences of real number such that $\lim_{n\to\infty}(x_n) = x$ and $\lim_{n\to\infty}(y_n) = y$, then show that $\lim_{n\to\infty}(x_n + y_n) = x + y$.

(b) Let (x_n) be a sequence of positive real numbers

such that $L = \lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right)$ exists. Show that if $L < 1$,

then (x_n) converges and $\lim_{n\to\infty}(x_n) = 0$.

(c) State Squeeze theorem and show that if

$z_n = \left(2^n + 3^n\right)^{\frac{1}{n}}$ then $\lim_{n\to\infty} z_n = 3$.

(d) Let $X = (x_n)$ be a sequence of real numbers defined by $x_1 = 1$ and

$$x_{n+1} = \sqrt{2 + x_n} \quad \text{for } n \in \mathbb{R}.$$

Show that the sequence (x_n) is convergent and find its limit.

4. (a) Prove that if a sequence (x_n) is a monotone decreasing and bounded below sequence of real numbers, then it is convergent.

(b) State Bolzano Weierstrass Theorem for Sequences. Show that the sequence $((-1)^n)$ is divergent.

(c) Find limit inferior and limit superior of the following sequences:

(i) $\left(\sin\left(\frac{n\pi}{4}\right)\right)$

(ii) $(3 + (-1)^n)$

(d) Show that every Cauchy sequence of real numbers is bounded. Is the converse true? Justify your answer.

(ii) Find the interval in which the function is concave up or concave down.

(b) Determine the critical points of the function $f(x) = \frac{(x+1)^2}{x^2+1}$. Also, find the horizontal and vertical asymptotes.

(c) Trace the graph of the function given by

$$f(x) = x^4 - 4x^3 + 10.$$

6. (a) Determine the intervals where the function given by $f(x) = 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$ is concave up or concave down. Also, find the points of inflection, if any.

(b) Find the critical points of $f(x) = x^{\frac{1}{3}}(x-4)$. Find the intervals on which f is increasing or decreasing. Also, find the local extreme values of the function.

(c) Find the oblique asymptotes of the curve given by the equation

$$4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 - y^2 + 2xy = 7$$

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 3553 **I**

Unique Paper Code : 2354001001

Name of the Paper : GE : Fundamentals of Calculus

Name of the Course : **Common Prog. Group**

Semester : I

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory and carry equal marks.
3. This question paper has **six** questions.
4. Attempt any **two** parts from each question.
5. Use of Calculator not allowed.

1. (a) Evaluate the following limits.

(i) $\lim_{x\to 0} x^2 \sin\frac{1}{x^2}$

(ii) $\lim_{x\to 0} \frac{3\sin^{-1}x}{4x}$

(b) Examine the continuity and differentiability of the function

$$f(x) = \begin{cases} x^2 \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(c) Find the n^{th} derivative of $y = \tan^{-1} x$.

2. (a) If $y = \left(x + \sqrt{1+x^2}\right)^m$, find the value of y_n at $x = 0$.

(b) If $u = f(r)$, where $r^2 = x^2 + y^2$,

show that : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

(c) If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, $x \neq y$,

show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$ and

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$$

3. (a) State Rolle's theorem.

Show that there is no real number t for which the equation $x^3 - 3x + t = 0$ has two distinct solutions in the interval $[-1, 1]$.

(b) State Lagrange's Mean value theorem. Use it to prove that

$$\frac{x-1}{x} < \log x < x - 1 \text{ for } x > 1.$$

(c) State and prove Cauchy's Mean value theorem.

4. (a) State Taylor's theorem with Lagrange's form of remainder.

Show that for any $k \in \mathbb{N}$

$$\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2k+1}}{2k+1} \text{ for } x > 0$$

(b) Evaluate $\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$.

(c) For which value of constants, a and b is it true that

$$\lim_{x \to 0} (x^{-3} \sin 2x + ax^{-2} + b) = 1$$

5. (a) For the function given by $f(x) = \frac{(x^2+1)}{(x^2-9)}$.

(i) Find the intervals in which the function is increasing or decreasing.

This question paper contains **3** printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **1565**

Unique Paper Code : **2352571101**

Name of the Paper : **DSC : Topics in Calculus**

Name of the Course : **B.A./B.Sc. (Prog.) with Mathematics as Non-Major/Minor**

Semester : **I**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt all questions by selecting *two* parts from each question.

All questions carry equal marks.

1. (*a*) Show that the function $f(x) = |x| + |x - 1|$ is not differentiable at $x = 0$ and $x = 1$.

(*b*) Find the *n*th differential coefficients of $\sin^5 x \cos^3 x$.

(*c*) If $u = \sin^{-1} \dfrac{x^2 + y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

2. (*a*) Examine the continuity of the function $f(x) = \dfrac{\sqrt{1 - \cos 2x}}{x}$, $x \neq 0$; $f(0) = 0$, $x = 0$.

In case of discontiuinty, state the kind of discontinuity.

(*b*) If $y = \left[x + \sqrt{1 + x^2}\right]^m$, find $y_n(0)$.

(*c*) If $= e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2h^2z^2)e^{xyz}$.

3. (a) Evaluate : $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$.

(b) Show that

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots\ldots\ldots\ldots + \frac{(-1)^{n-2}x^{n-1}}{n-1} + \frac{(-1)^{n-1}x^n}{n(1+\theta x)^n},\ 0 < \theta < 1.$$

(c) Verify whether Rolle's theorem is applicable to the function $f(x) = 2 + (x - 1)^{\frac{2}{3}}$ in the interval [0, 2] or not.

4. (a) Expand $e^{\sin x}$ as far as the term containing x^4.

(b) If in Cauchy's mean value theorem, $f(x) = \frac{1}{x^2}$, and $g(x) = \frac{1}{x}$, then show that c is the harmonic mean of a and b.

(c) Find a, b, c so that $\lim_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$.

5. (a) Find all the asymptotes of the curve :

$$x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0.$$

(b) Find the range of values of x in which the curve

$$y = 3x^4 - 4x^3 + 1$$

is concave upwards or downwards. Also find its points of inflexion.

(c) Find the reduction formula for

$$\int \sin^n x\, dx.$$

Hence, evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x\, dx$.

6. (*a*) Determine the position and nature of the double points on the curve :

$$x^3 + x^2 + y^2 - x - 4y + 3 = 0.$$

(*b*) Trace the curve $y = x(x^2 - 1)$.

(*c*) Obtain a reduction formula for $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$. Hence evaluate :

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, dx.$$

(c) Which of the following is a subspace of $\mathbb{R}^3$? Justify. (6)

(i) $W_1 = \{(a, b, c) \in \mathbb{R}^3 \mid a+b=c\}$

(ii) $W_2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2+b^2=c^2\}$.

6. (a) Define a linear transformation. Determine whether $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$T([a, b]) = [2a-3b,\ 3a+4b]$ is a linear transformation or not. (6.5)

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T([1,-1,0]) = [2,1]$, $T([0,1,-1]) = [-1,3]$ and $T([0,1,0]) = [0,1]$. Find $T([a, b, c])$ for any $[a, b, c] \in \mathbb{R}^3$. (6.5)

(c) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T([a, b, c]) = [a, 0, a-b+c]$. Find a basis for ker(T) and a basis for range(T). Also, verify the rank-nullity theorem. (6.5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 6463 **I**

Unique Paper Code : 42354302

Name of the Paper : Algebra

Name of the Course : **B.Sc. (Prog.) Physical Sciences/Mathematical Sciences**

Semester : III DSC

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. This question paper has **six** questions in all.
3. Attempt any **two** parts from each question.
4. **All** questions are compulsory.

1. (a) Let $G = \left\{\begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| a \in \mathbb{R},\ a \neq 0\right\}$. Show that G is an abelian group under matrix multiplication. (6)

(b) Describe the symmetries of an equilateral triangle and write the complete multiplication table for D_3. (6)

(c) Find orders of each of the elements of U(10). Show that it is cyclic and find all its generators. (6)

2. (a) State and prove Lagrange's theorem. Give statement of Fermat's little Theorem. (6.5)

(b) Prove that intersection of two subgroups is a subgroup. Is the union of two subgroups a subgroup? Justify your answer. (6.5)

(c) Define center of a group. Prove that center of a group G is a subgroup of G. (6.5)

3. (a) Find the order of each of the following permutations : (6)

(i) (124)(357)

(ii) $\begin{bmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,5\,4\,6\,3 \end{bmatrix}$

(b) Let H be a subgroup of group G, and let a and b belong to G. Prove that aH = H if and only if $a \in H$. (6)

(c) Define a normal subgroup. Let

$H=\left\{\begin{bmatrix} a & b \\ 0 & d\end{bmatrix} : a,b,d \in \mathbb{R}, ad \neq 0\right\}$. Is H a normal subgroup of GL(2, $\mathbb{R}$)? Justify. (6)

4. (a) Let R be a ring with unity 1 and a is an element of R such that $a^2 = 1$. Let $S = \{ara: r \in R\}$. Prove that S is a subring of R. Does S contain 1? (6.5)

(b) Show that $\mathbb{R}\oplus\mathbb{R}$ is not an Integral Domain. (6.5)

(c) Let $S = \{a + ib: a, d \in \mathbb{Z}, b \text{ is even}\}$. Show that S is not an ideal of $\mathbb{Z}[i]$. (6.5)

5. (a) Determine whether or not the set $S=\left\{\begin{bmatrix} 2 & 1 \\ 1 & 0\end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2\end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix}\right\}$ is linearly independent over Z_5. (6)

(b) Find a basis and dimension for the subspace W of $\mathbb{R}^3$ defined by $W = \{(a, b, c) \in \mathbb{R}^3 \mid 2a - 3b + c = 0\}$. (6)

(b) Simplify the polynomial

$$p = (a + b)(a+c) + abc$$

using Karnaugh diagram. (7.5)

(c) Find all prime implicants of $xy'z + x'yz' + xyz' + xyz$ and form the corresponding prime implicant table. (7.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1160 **I**

Unique Paper Code : 2352012303

Name of the Paper : Discrete Mathematics (DSC-9)

Name of the Course : **B.Sc. (H) Mathematics**

Semester : III

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.

2. All questions are compulsory.

3. Questions 1 to 6 have three parts each. Attempt any **TWO** parts from each Question. Each part carries 7.5 marks.

4. Use of the Calculator is not allowed.

1. (a) Suppose $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,\ldots\}$. Define $m \le n$ in $\mathbb{N}_0$, if and only if there exists $k \in \mathbb{N}_0$, such that $n = km$. Prove that $(\mathbb{N}_0, \le)$ is a partially ordered set. Is $(\mathbb{N}_0, \le)$ a chain, an antichain or none? Justify your answer. (7.5)

(b) Define when we say that the two sets have the same cardinality. Prove that the sets $(0,1)$ and (a, ∞) have the same cardinality. (7.5)

(c) Draw the Hasse diagrams for the following ordered sets:

(i) $(\wp(X), \subseteq)$, with $X = \{1,2,3\}$. Here $\wp(X)$ is the power set of X.

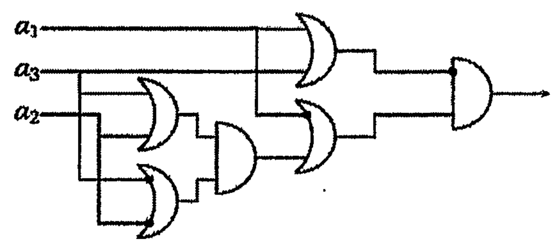
(ii) Dual of $\oplus\ M_3$, where $M_n = \mathbf{1} \oplus \bar{\mathbf{n}} \oplus \mathbf{1}$.

(b) Show that (B, gcd, lcm) is a Boolean algebra if B is the set of all positive divisors of 110. (7.5)

(c) Find Disjunctive Normal form and Conjunctive Normal form of the following Boolean polynomial f:

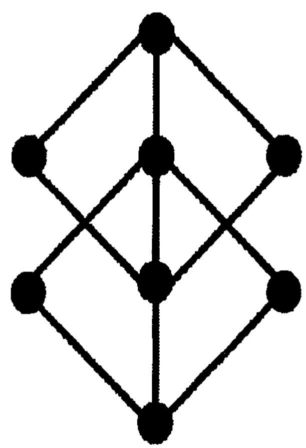
$$f = a(b+c)' + (ab + c')a. \quad (7.5)$$

6. (a) Determine the Boolean polynomial p of the circuit



and then draw its contact diagram. (7.5)

(ii) State M_3 - N_5 theorem. Apply the theorem to find if the lattice L shown below is modular or distributive. (3,4.5)



5. (a) Let $f: \mathbb{B}^3 \rightarrow \mathbb{B}$ have the value 1 precisely at the arguments (0,0,0), (0,1,0), (0,1,1), (1,0,0). Find a Boolean polynomial p for this function and simplify p. (7.5)

(iii) $\mathbf{2} \times \mathbf{3}$

Here n denotes the chain obtained by giving the set $P = \{0,1,\ldots, n\text{-}1\}$, the order in which $0 < 1 < \ldots < n-1$ and $\bar{\mathbf{n}}$ for P regarded as an antichain. (3, 2.5, 2)

2. (a) Define maximal element of an ordered set. Give an example of an ordered set which has exactly one maximal element but does not have a greatest (or maximum) element. Give one example of an ordered set with exactly 3 maximal elements. (7.5)

(b) (i) State duality principle in ordered sets.

(ii) Define an order preserving map between two ordered sets and prove that the composite map of two order preserving maps is order preserving. (3, 4.5)

(c) Define bottom and top element in an ordered set. Give one example of an ordered set in which bottom and top both exist and one example in which none of them exist. (7.5)

3. (a) (i) Prove that in a lattice L, for any a, b, c, $d \in L$, $a \leq c$, $b \leq d$ implies that $a \vee b \leq c \vee d$.

(ii) Prove that a lattice L is a chain if and only if every nonempty subset of L is a sublattice of L. (3, 4.5)

(b) Prove that in any lattice L, the following holds

$$((x \wedge y) \vee (x \wedge z)) \wedge ((x \wedge y) \vee (y \wedge z)) = x \wedge y, \text{ for all } x, y, z \in L. \quad (7.5)$$

(c) Let L and K be lattices and $f: L \to K$ be a map. Show that the following are equivalent:

(i) f is order preserving.

(ii) $(\forall a, b \in L), f(a \wedge b) \leq f(a) \wedge f(b)$. (7.5)

4. (a) Let L and M be two ordered sets and f be an order isomorphism from L onto M. Prove that if L is a lattice, then M is also a lattice, and f is a lattice isomorphism. (7.5)

(b) Prove that the direct product L x K of two distributive lattices L and K is also a distributive lattice. (7.5)

(c) (i) Prove that every sublattice of a modular lattice L is modular.

(c) The set $G = \{1,4,11,14,16,19,26,29,31,34,41,44\}$ is a group under multiplication modulo 45. Write G as an external and an internal direct product of cyclic groups of prime-power order. (7.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1083 **I**

Unique Paper Code : 2352012301

Name of the Paper : Group Theory

Name of the Course : **B.Sc. (H) Mathematics UGCF**

Semester : III – DSCC-7

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator not allowed.

1. (a) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles. (7.5)

(b) (i) Let S_n denote the symmetric group of degree n. In S_3, find elements α and β such that $|\alpha| = 2$, $|\beta| = 2$ and $|\alpha\beta| = 3$.

(ii) Let $\beta \in S_7$ and $\beta^4 = (2\ 1\ 4\ 3\ 5\ 6\ 7)$. Then find β. (3,4.5)

(c) (i) Give two reasons to show that the set of odd permutations in S_n is not a subgroup of S_n.

(ii) Define even and odd permutations and show that the set of even permutations in S_n is a subgroup of S_n. (3, 4.5)

(c) Without doing any calculations in $\mathrm{Aut}(\mathbb{Z}_{105})$, determine how many elements of $\mathrm{Aut}(\mathbb{Z}_{105})$ have order 6. (7.5)

6. (a) For any group G, prove that $G/Z(G) \cong \mathrm{Inn}(G)$. (7.5)

(b) Define the internal direct product of a collection of subgroups of a group G. Let $\mathbb{R}^*$ denote the group of all nonzero real numbers under multiplication. Let $\mathbb{R}^+$ denote the group of all positive real numbers under multiplication. Prove that $\mathbb{R}^*$ is the internal direct product of $\mathbb{R}^+$ and the subgroup $\{1, -1\}$. (7.5)

(c) If ϕ is an onto homomorphism from group G_1 to group G_2, then prove that $G_1/\text{Ker}\,\phi$ is isomorphic to G_2. Hence show that if G_1 is finite, then order of G_2 divides the order of G_1. (7.5)

5. (a) Let G be a group and let $a \in G$. Define the inner automorphism of G induced by a. Show that the set of all inner automorphisms of a group G, denoted by Inn(G), forms a subgroup of Aut(G), the group of all automorphisms of G. Find $\text{Inn}(D_4)$. (7.5)

(b) Prove that the order of an element in a direct product of a finite number of finite groups is the lcm of the orders of the components of the element, i.e., $|(g_1, g_2, \ldots, g_n)| = \text{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$. Also, find the number of elements of order 7 in $\mathbb{Z}_{49} \oplus \mathbb{Z}_7$. (7.5)

2. (a) (i) Let a be an element in a group G such that $|a| = 15$. Find all left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$.

(ii) State and prove Lagrange's theorem.

(3,4.5)

(b) Suppose that G is a group with more than one element and G has no proper, non-trivial subgroups. Prove that $|G|$ is prime. (7.5)

(c) Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication and let $H = \{a + bi \in \mathbb{C}^* \mid a^2 + b^2 = 1\}$. Give a geometrical description of the coset $(3 + 4i)H$. Give a geometrical description of the coset $(c + di)H$. (7.5)

3. (a) (i) Let G be a group and H be its subgroup. Prove that if H has index 2 in G, then H is normal in G.

(ii) If a group G has a unique subgroup H of some finite order, then show that H is normal in G. (3,4.5)

(b) (i) Prove that a factor group of a cyclic group is cyclic. Is converse true? Justify your answer.

(ii) Let G be a group and let Z(G) be the center of G. If G/Z(G) is cyclic, then show that G is Abelian. (3,4.5)

(c) (i) Let ϕ be a group homomorphism from group G_1 to group G_2 and H be a subgroup of G_1. Show that if H is cyclic, then $\phi(H)$ is cyclic.

(ii) How many homomorphisms are there from $\mathbb{Z}_{20}$ to $\mathbb{Z}_8$? How many are there onto $\mathbb{Z}_8$? (3,4.5)

4. (a) (i) Suppose that ϕ is a homomorphism from U(30) to U(30) and Ker ϕ = {1,11}. If $\phi(7) = 7$, find all the elements of U(30) that map to 7.

(ii) Let ϕ be a homomorphism from a group G_1 to group G_2. Show that $\phi(a) = \phi(b)$ iff aKer ϕ = bKer ϕ. (3,4.5)

(b) (i) Is U(8) isomorphic to U(10)? Justify your answer.

(ii) Show that any infinite cyclic group is isomorphic to the group of integers under addition. (3,4.5)

This question paper contains 4 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **6294**

Unique Paper Code : **42344304**

Name of the Paper : **Operating Systems**

Name of the Course : **B.Sc. Mathematical Sc./B.Sc. Programme (CBCS-LOCF)**

Semester : **III**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All parts of Question 1 (Section A) are compulsory.

Attempt any *five* questions from Section B.

All questions in Section B carry equal marks.

Attempt *all* parts of a question together.

Section–A

1. (*a*) How does an Operating System prevent the CPU from being infinitely used by a user program ? 2

(*b*) What do you understand by "Convoy Effect" with respect to CPU scheduling ? 2

(*c*) Why do programmers use APIs over actual System Calls for reading/writing data on devices ? 2

(*d*) List two system calls of an Operating System, each for Process Management and File System Management. 2

(e) What is the need of having secondary storage in the computer system ? 2

(f) What is the "degree of multiprogramming" ? Which scheduler controls the degree of multiprogramming ? 2

(g) What is the use of a page table in paging memory management ? 2

(h) Assuming a 1-KB page size, find the page number and offset for the address reference 42095 given in decimal number. 2

(i) What is an absolute pathname ? Also give an example. 2

(j) What will be following Unix/Linux commands give as output ? (Here f1, f2, f3, f4 are data files) :

(i) cal 5 2024 1

(ii) cat f1 f2 f3 > f4 1

(iii) tail –n 3 f1 1

(k) Consider a logical address space of 64 pages with 1024 words per page, mapped onto a physical memory of 32 frames :

(i) How many bits are required in the Logical Address ? 2

(ii) How many bits are required in the Physical Address ? 2

Section–B

(Attempt any *five*)

2. (a) What is dual-mode operation of an Operating System ? Explain with the help of diagram. 6

(b) List the *two* advantages and *two* disadvantages of layered approach to system design. 4

3. (*a*) Define process. Explain different process states with the help of a diagram. 6

(*b*) How many processes are created by the following code ? Justify your answer : 4

```
int main()
{
 int i;
 for (i = 0; i < 4; i++)
     fork();
   return 0;
}
```

4. Suppose the following processes arrive for execution at the times indicated below :

Process	Arrival Time	Burst Time (ms)
P1	0.0	8
P2	0.4	4
P3	1.0	1

(*i*) Draw Gantt charts that illustrate the execution of these processes using the following scheduling algorithms :

Non-Preemptive SJF, Preemptive SJF and RR (quantum = 2). 6

(*ii*) Compute average turnaround time of each process for the Preemptive SJF and Non-Preemptive SJF scheduling algorithms. 2

(*iii*) Compute average waiting time of each process for the Non-Preemptive SJF and RR (quantum=2) scheduling algorithms. 2

5. (*a*) Consider a paging system with the page table stored in memory where a memory reference takes 100 ns.

(*i*) How long does a paged memory reference take ? 2

(*ii*) If we add TLBs, and 80% of all page table references are found in the TLBs, what is the effective memory access time ? (Assume that finding the page-table entry in the TLBs takes 0 ns, if the entry is there). 4

(*b*) What is external fragmentation ? Explain how can it be reduced ? 4

6. (*a*) Under what circumstances does a page fault occur ? Describe the steps to handle page fault. Also draw the diagram. 6

(*b*) Explain tree structured directory with a suitable example. 4

7. (*a*) Given memory partitions of sizes 500 KB, 300 KB, 100 KB, 200 KB and 600 KB (in order). How would each of the first fit, best fit and worst fit algorithms place the processes of 250 KB, 350 KB, 100 KB and 426 KB (in order) ? Which algorithm makes the most efficient use of memory and why ? 8

(*b*) Name *two* services provided to user by an Operating System. 2

8. (*a*) Write Unix/Linux commands for the following : 6

(*i*) Count number of occurrences of the word "Unix" in a file F1.

(*ii*) Sort the file F2 alphabetically.

(*iii*) Display date in the mm/dd/yy format.

(*iv*) Copy file "hello.txt" from parent directory to the current directory.

(*v*) Give read permission to owner of a file F3.

(*vi*) Cut the first field of the output of ls –1.

(*b*) Write a shell script to compute factorial of a number N. 4

(b) Prove that the systematic sampling will be more efficient than SRSWOR if the intra class correlation coefficient $\rho < -\frac{1}{N-1}$. (Notations have their usual meaning). (8,7)

8. Write short note :

(i) Stratified Random Sampling

(ii) Probability and non-probability sampling

(iii) Steps in conducting sample surveys (5×3)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1087 **I**

Unique Paper Code : 2372012301 (NEP-UGCF)

Name of the Paper : Sample Surveys (DSC)

Name of the Course : **B.Sc. (Hons.) STATISTICS**

Semester : III

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **six** questions in all.
3. **All** questions carry equal marks.
4. Use of non-programmable scientific calculator is allowed.

1. (a) Define Simple random sampling. Derive the variance of the sample mean based on SRSWOR. Also, derive the unbiased estimator of this variance.

(b) From a simple random sample of size n drawn from N units by SRSWOR, a simple random sub-sample of ni units is duplicated and added to the original sample. Show that the mean based on $(n + n_1)$ units is an unbiased estimator of the population mean. Also, obtain its variance. How does it compare with the variance of the estimator based on n units only? (7,8)

2. (a) If (X_i, Y_i) is the pair of the variables defined for every unit (i = 1,2,...,N) of the population and $\overline{y}_n$ and $\overline{x}_n$ are the corresponding sample means of a

(b) Define Difference estimator and derive from it the regression estimator. Also obtain the variance of regression estimator under first approximation. (6,9)

7. (a) For a given cost function $C = c_1n_1 + c_2n_2$, derive $\mathrm{Var}(\overline{y}_{st})$ under optimum allocation. Further, assume that a sampler has two strata with relative sizes P_1 and P_2 and equal population standard deviations S_1 and S_2. He prefers to use proportional allocation but does not wish to incur a substantial increase in variance as compared with optimum allocation. Show that (ignoring fpc):

$$\frac{\mathrm{Var}(\overline{y}_{st})_{prop}}{\mathrm{Var}(\overline{y}_{st})_{opt}} = \frac{P_1c_1 + P_2c_2}{\left(P_1\sqrt{c_1} + P_2\sqrt{c_2}\right)^2}$$

Compute the sample sizes n_1, n_2 in the 2 strata minimizing the total size of sample in each case under the following conditions (ignore the fpc):

(i) The standard error of the estimated population mean is to be 0.1.

(ii) The standard error of the difference between the two estimated stratum means is to be 0.1.

(iii) The standard error of the estimated mean of each stratum is to be 0.1. (6,9)

6. (a) A population of N units contains N_1 units belonging to class A and N_2 $(N-N_1)$ units belonging to class B. Obtain an unbiased estimator of N_1 and unbiased estimator of its variance, under simple random sampling without replacement.

simple random sample of size n selected without replacement, then prove that

$$\operatorname{cov}(\bar{x}_n, \bar{y}_n) = \frac{N-n}{n(N-1)}\operatorname{cov}(x,y) .$$

Hence prove that $r(\bar{x}_n, \bar{y}_n) = \rho(X,Y)$. (Notations have their usual meaning).

(b) Derive the condition under which a systematic sampling has the same precision as the corresponding stratified random sampling with one unit per stratum. (8,7)

3. (a) Compare stratified random sampling under Neyman and proportional allocations with simple random sampling without replacement.

(b) State the practical difficulties in adopting Neyman method of allocation of a sample to different strata. How much would be the increase in variance, on the average, if the allocation is based on the estimates of strata mean squares? Also, compare it with proportional allocation. (6,9)

4. (a) (i) In SRSWOR, prove that the probability of selecting a specified unit of the population at any given draw is equal to the probability of selecting it at the first draw.

(ii) If a random sample of size n is drawn from a population of size N by SRSWOR, then prove that covariance between any two members of the sample is $-\frac{\sigma^2}{N-1}$.

(b) Obtain the bias in the ratio estimator to the first approximation. Also show that this bias vanishes if the regression line is linear and passes through origin. (3,4,8)

5. (a) Explain the Yates' end corrections in systematic sampling for a population with a linear trend. Hence show that the sample mean obtained after applying the end corrections overlaps the population mean.

(b) In a stratification with 2 strata, the values of the P_i and S_i are as follows :

Stratum	P_i	S_i
1	0.8	2
2	0.2	4

This question paper contains 3 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : 5774

Unique Paper Code : 32353301

Name of the Paper : LaTeX and HTML

Type of the paper : SEC

Semester : III

Name of the Course : B.Sc. (Hons) Mathematics

Duration : 2 Hours Maximum Marks : 38

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt all questions.

Part of the questions to be attempted together.

Use of calculator is not allowed.

1. Fill in the blanks : $8\times1=8$

(*i*) The command .... is used to emphasize text in LaTeX.

(*ii*) The symbol ∞ can be produced in LaTeX using the command ....

(*iii*) .... command is used to create horizontal dots above the line in LaTeX.

(*iv*) The command to produce name of institute in a beamer presentation is ....

P.T.O.

(v) .... command is used in pstricks to put a label at a specified point.

(vi) The HTML element is closed with .... tag.

(vii) Boldface text on a webpage is obtained with the .... element.

(viii) .... tag is used in HTML to add the largest heading to a paragraph.

2. Answer any *five* parts from the following : 5×2=10

(i) Write the output of the command :

`$\frac{d} {dx}\left(\int_{0}^{x}f(t)\,dt\right)=f(x)$.`

(ii) Write a set of commands to be put in the main document in LaTeX to produce :

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1.$$

(iii) Write a code in LaTeX to get the following :

$$\begin{bmatrix} 5 & 9 & 7 \\ 4 & 2 & 8 \\ 6 & 2 & 3 \end{bmatrix}$$

(iv) Write the following postfix expressions in the standard form :

sin 1 add 2 exp 1 x sub div.

(v) Write the command in PSTricks to plot the function $y = \sin x$.

(vi) What is the use of <a> tag in HTML ? Give its general syntax.

(vii) What is wrong with the following HTML construction ?

<p> This is   *bold and italics* </p>

3. Answer any *four* parts from the following : 4×5=20

(*i*) Write the code to typeset the following :

$$|x-3| = \begin{cases} x-3 & x \geq 3 \\ -x+3 & x < 3 \end{cases}$$

(*ii*) Write a code in LaTeX to get the following :

$$\begin{aligned} (a+b)^2 &= (a+b)\,(a+b) \\ &= a\,(a+b) + b\,(a+b) \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

(*iii*) Write a code to make a beamer presentation of 5 pages (including title and thank you page) on any topic with diagram/picture.

(*iv*) Draw a circle with shaded sector.

(*v*) Write an HTML code to generate web page :

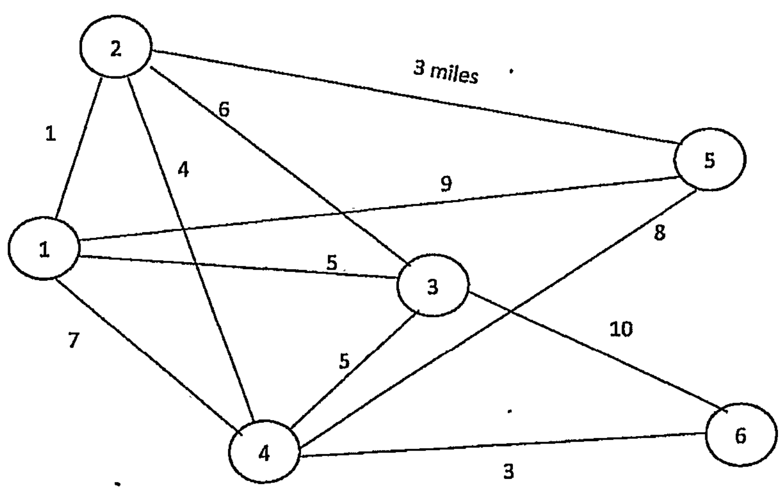
University of Delhi

- Skill Enhancement Courses (SEC):
  1. Sec-1 : LaTeX and HTML
  2. Sec-2 : Computer Algebra System
- Discipline Specific Elective (DSE):
  1. DSE-1 : Numerical Methods
  2. DSE-2 : Discrete Mathematics

What is the average waiting time per customer? What is the percentage idle time of the facility? [Assume that the system starts at t=0]. (8)

(b) What is a fixed charge transportation problem? Give the formulation of the fixed charge transportation problem. (7)

7. Consider the following network where the numbers on links represents actual distance between the corresponding nodes. Find the minimal spanning tree. (15)



[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 6266 **I**

Unique Paper Code : 42363307

Name of the Paper : Operational Research Applications (SEC)

Name of the Course : **B.Sc. (Prog) Mathematical Sciences CBCS**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **five** questions.
3. **All** questions carry equal marks.
4. Show your working clearly in your answer sheet.

1. (a) The marketing Department of Everest Company has collected information on the problem of advertising for its products. This relates to the advertising media available. The number of families expected to be reached with each alternatives, cost per advertisement the maximum

availability of each medium and the expected exposure of each one (Measured as the relative value of one advertisement in each of the media).

The information is given as under :

Advertising media	No. of families to cover	Cost/ad. (Rs.)	Maximum availability (No. of times)	Expected exposure (Unit)
TV (30 sec)	3000	8000	8	80
Radio (15 sec)	7000	3000	30	20
Sunday Edition (1/4 page)	5000	4000	4	50
Magazine (1 page)	2000	3000	2	60

Other information and requirements :

(a) The advertising budget is Rs. 7000.

(b) At least 40,000 families should be covered.

(c) At least 2 insertions be given in Sunday Edition of Daily but not more than 4 advertisements should be given on T.V.

Formulate this as a linear programming problem to maximize the expected exposure. (10)

(b) Describe media allocation problem and write a general mathematical formulation of the model. (5)

Which items and what quantities should be loaded on the plane so as to maximize the value of the cargo transported? Formulate this as an IPP and find the solution. (8)

(b) Consider the following production data :

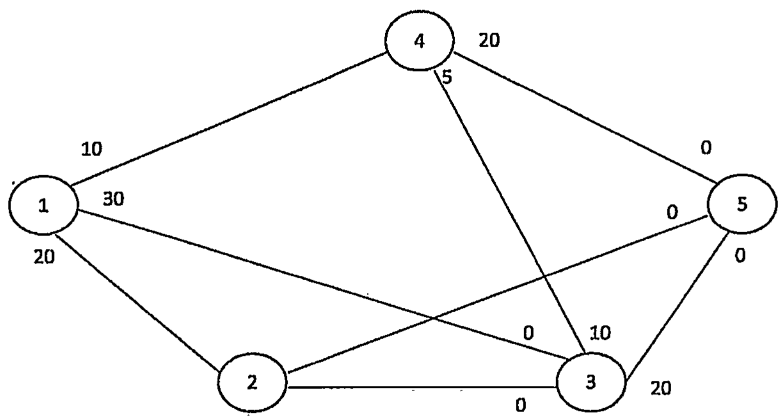
Product	Profit per Unit (Rs)	Direct Labour Requirement (hours)
1	6	12
2	8	11
3	5	14

Fixed Cost (Rs.)	Direct Labour Requirement
8,000	up to 10,000 hours
16,000	10,000-20,000 hours
22,000	20,000-30,000 hours

Formulate an integer programming problem to determine the production schedule so as to maximize the net profit. (7)

6. (a) Customer arrive at a milk booth for the required service. Assume that inter-arrival and service times arc constant and given by 1.8 and 4 time units, respectively. Simulate the system for 14 time units.

(b) In the following network, the bi-directional capacities arc shown on the links using the normal convention. For example the link 3-4 the flow limit is 10 units from node 3 to node 4 and it is 5 units from node 4 to node :



Illustrate three cuts on the network flows and obtain their capacities. (7)

5. (a) Four items are considered for loading on an aero plane, which has a capacity to load upto 25 tonnes. The weights and values of the items are as follows :

Item	:	A	B	C	D
Weight (tonnes)	:	2	7	5	3
Value (per unit)	:	10	36	25	14

2. (a) An automobile manufacturer produces cars and trucks in a factory that is divided into two shops. Shop A which performs the basic operations must work 5 man- days on each truck but only 2 man-days on each car. Shop B which performs finishing operations must work 3 man-days for each car or truck that it produces. Because of men and machine limitations, shop A has 180 man days per week available, while shop B has 135 man-days per week. If the manufacturer makes a profit of Rs.300 on each truck and Rs.200 on each car, how many of each should be produced to maximize profit of the organization. Formulate the linear programming model for this problem. (8)

(b) Distinguish between linear programming problem and integer programming problem. (7)

3. (a) Reliance company is considering four possible investment opportunities. The following table present information about the investment (in Rs. thousand) profits:

Project	Present value of expected Return	Capital Required Year-wise by Projects		
1	650	700	550	400
2	700	850	550	350
3	225	300	150	100
4	250	350	200	-
Capital available for investment		1200	700	400

In addition, project 1 and 2 are mutually exclusive and project 4 is contingent on the prior acceptance of project 3. Formulate an integer programming model and solve it to determine which project should be accepted and which should be rejected in order to maximize the present value from the accepted project. (8)

(b) The SUPERSUDS CORPORATION is developing its marketing plans for next year's new products. For three of these products, the decision has been made to purchase a total of five TV spots for commercials on national television networks. The problem we will focus on is how to allocate the five spots to these three products, with a maximum of three spots (and a minimum of zero) for each product. Use cutting method to solve the problem.

Number of TV Spots	Profit		
	1	2	3
0	0	0	0
1	1	0	-1
2	3	2	2
3	3	3	4

(7)

4. (a) SOUTHWESTERN AIRWAYS needs to assign its crews to cover all its upcoming flights. We will focus on the problem of assigning three crews

based in San Francisco to the flights listed in the first column of Table 12.4. The other 12 columns show the 12 feasible sequences of flights for a crew. (The numbers in each column indicate the order of the flights.) Exactly three of the sequences need to be chosen (one per crew) in such a way that every flight is covered. (It is permissible to have more than one crew on a flight, where the extra crews would fly as passengers, but union contracts require that the extra crews would still need to be paid for their time as if they were working.) The cost of assigning a crew to a particular sequence of flights is given (in thousands of dollars) in the bottom row of the table. The objective is to minimize the total cost of the three crew assignments that cover all the flights.

	Feasible Sequence of Flights											
Flight	1	2	3	4	5	6	7	8	9	10	11	12
1.San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago		2						2			2	
10. Seattle to San Francisco			2					4	4			5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

(8)

5. (a) Let $f(x) = |x| + |x + 1|$, $x \in \mathbb{R}$. Show that f is not differentiable at $x = 0$ and $x = -1$. (6)

(b) Let f and g be differentiable at $c \in \mathbb{R}$. Show that fg is also differentiable at c and $(fg)' = f(c)g'(c) + f'(c)g(c)$. Hence find the derivative of $x^3\sin(x)$. (6)

(c) State and prove the Mean Value theorem. (6)

6. (a) Let f be defined on an open interval containing x_0 such that f assumes its maximum/minimum at x_0. If f is differentiable at x_0, then prove that $f'(x_0) = 0$. (7½)

(b) Find Taylor's series expansion of cos (x) about $x = 0$. (7½)

(c) Define a convex function on and interval $I \subseteq \mathbb{R}$. Check which of the following functions are convex :

(i) $|x|$, $x \in [-1,1]$

(ii) $x + \sin x$, $x \in [0,\pi]$ (7½)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5850 **I**

Unique Paper Code : 32351301

Name of the Paper : Theory of Real Functions

Name of the Course : **B.Sc. (H) Mathematics (LOCF)**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1 (a) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{if } x \text{ is irrational}\end{cases}$$

Show that f does not have a limit at any $c \in \mathbb{R}$. (6)

(b) Use the $\in - \delta$ definition of the limit to show that

$$\lim_{x\to 3}\frac{2x+3}{4x-9}=3. \qquad (6)$$

(c) Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x\to c} f = \iota$ and $\lim_{x\to c} g = m$, then show that $\lim_{x\to c}(f-g) = l-m$. (6)

2. (a) State Squeeze Theorem and use it to show that

$$\lim_{x\to 0}\frac{\cos x-1}{x}=0. \qquad (6)$$

(b) Let f: $\mathbb{R} \to \mathbb{R}$ be such that $f(x + y) = f(x) + f(y)$ for all x, $y \in \mathbb{R}$. Assume that $\lim_{x\to 0} f = \iota$ exists. Prove that $\iota = 0$, and then prove that f has a limit at every $c \in \mathbb{R}$. (6)

(c) Use the definition of infinite limits to show that

$$\lim_{x\to 1^+}\frac{x}{x-1}=\infty, \ (x \neq 1). \qquad (6)$$

3. (a) Let f: $[a, b] \to \mathbb{R}$ be a continuous function such that for each $x \in [a, b]$ there exists $y \in [a, b]$ with $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there exists $c \in [a, b]$ with $f(c) = 0$. (6)

(b) Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$ but that it is not uniformly continuous on $[0, \infty)$. (6)

(c) Let f: $[a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Let $w = \inf \{x \in [a, b] : f(x) < 0)$. Show that $f(w) = 0$. (6)

4. (a) Using Mean value theorem, show that if $\alpha > 1$, then $(1 + x)^{\alpha} \geq 1 + ax$ for all $x > -1$ with equality if and only if $x = 0$. (6)

(b) Let f be a real valued differentiable function on [a, b] and there exists $k \in \mathbb{R}$ such that $f'(a) < r < f'(b)$. Show that $f'(c) = r$ for some $c \in (a, b)$. (6)

(c) Let I, J be intervals in $\mathbb{R}$ and f: $I \to \mathbb{R}$, g: $J \to \mathbb{R}$ be two functions such that $f(I) \subseteq J$. Let $c \in 1$. If f is differentiable at c and g is differentiable at $f(c) \in J$, then $g \circ f$ is differentiable at c and $g \circ f'(c) = g' \circ f(c).f'(c)$. (6)

(b) If f is a one-to-one continuous mapping of a compact metric space (X, d_X) onto a metric space (Y, d_Y), then show that f^{-1} is continuous on Y and hence, f is a homeomorphism of (X,d_X) onto (Y, d_Y). (6.5)

(c) If the real-valued function f is continuous on the closed bounded interval [a, b], then prove that f is uniformly continuous on [a, b]. (6.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5832 **I**

Unique Paper Code : 32351501

Name of the Paper : BMATH511 – Metric Spaces

Name of the Course : **B.Sc. (Hons.) Mathematics (LOCF)**

Semester : V, Core

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. In the question paper, given notations have their usual meaning unless and until stated otherwise.

1. (a) (i) Define a metric space. Show that the non-negativity metric axiom is implied by the other metric axioms of coincidence, symmetry and triangle inequality.

(ii) Let X be a non-empty set.

Define for x, y ε X

$$d(x,y)=\begin{cases}0 & \text{if } x=y\\ \sqrt{2} & \text{if } x\neq y\end{cases}$$

Prove that d a metric on X. (3+3)

(b) (i) Find the open ball in $C_R[0, 1]$ centred at $f(t)=t^2$ of radius 1. Give its geometrical interpretation.

(ii) Show that (3,5) is an open ball in R. Find its centre and radius. (4+2)

(b) Let (X, d_x) be a connected metric space and $f: (X, d_X) \rightarrow (Y, d_Y)$ be a continuous mapping. Then show that the space f(X) with the metric induced from Y is connected. (6)

(c) Prove that the intervals are connected sets in (R,d), the metric space of real numbers with the usual metric. (6)

6. (a) Let (X, d) be a metric space. Then prove that the following statements are equivalent:

(i) (X, d) is compact;

(ii) every collection of closed sets in (X, d) with empty intersection has a finite subcollection with empty intersection;

(iii) every collection of closed sets in (X, d) with the finite intersection property has nonempty intersection. (6.5)

(c) Let T be a continuous mapping of a complete metric space into itself. Define T^n inductively by $T^1 = T$ and $T^{n+1} = T \circ T^n$. If T^k is a contraction for some positive integer k, then the equation $Tx = x$ has one and only one solution. Moreover, for any $x \in X$, the sequence $\{T^n x\}_{n \geq 1}$ converges to that solution. (6.5)

5. (a) Let (X,d) be a metric space. Then show that the following statements are equivalent:

(i) (X,d) is disconnected;

(ii) There exist two nonempty disjoint subsets A and B, both open in X, such that $X = A \cup B$;

(iii) There exist two nonempty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(iv) There exists a proper subset of X that is both open and closed in X. (6)

(c) Suppose $X = C_R[a, b]$, the set of all continuous functions from [a,b] to $\mathbb{R}$ and $d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]; a<b\}$ be the associated metric. Then prove that (X, d) is a complete metric space. (6)

2. (a) (i) Suppose (X, d) is a metric space and F is a finite subset of X. Prove that the derived set F' of F is empty.

(ii) Let A be a finite subset of $\mathbb{R}$ with usual metric. Is A open in $\mathbb{R}$? Justify your answer. If not, then give an example of a metric space in which a finite set may be open.

(2.5+4)

(b) (i) Consider the subspace $X = I \cup J$ of R, where $I =]0,1[$ and $J = [4,7[$. Show that $d(\sup_x I, J) > 0$.

(ii) Find the interior and derived sets of Z and Q in R with usual metric. Determine which one of these sets is open or closed or none in R. Justify your answer. (2.5+4)

(c) (i) Is $Q \cup \{\sqrt{2}\}$ dense in R with usual metric? Justify your answer.

(ii) Give an example of a set which is open in a subspace Y of a metric space (X,d) but not open in X. Under what condition every open set in a subspace Y of a metric space (X,d) is open in X? Justify your answer.

(2+4.5)

3. (a) Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. (6)

(b) Define a uniformly continuous function on a metric space. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous, whereas the function $f: [0, 1] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous. (6)

(c) Let f be an isometry between metric spaces (X_1, d_1) and (X_2, d_2). Show that f is a homeomorphism between X_1 and X_2. Is the converse true? Justify your answer. (6)

4. (a) Show that the metrics d_1, d_2 and d_∞ defined on $\mathbb{R}^n$ by

$$d_1(x,y) = \textstyle\sum_{i=1}^{n}|x_i - y_i|\,;$$

$$d_2(x,y) = \left(\textstyle\sum_{i=1}^{n}|x_i - y_i|^2\right)^{\frac{1}{2}};\ \text{and}$$

$$d_\infty(x,y) = max\,\{|x_i - y_i| : 1 \le i \le n\}$$

are equivalent. (6.5)

(b) Let $X = [0, 2\pi] \subset R$ and $Y = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle in the complex plane. Consider the map $f : X \to Y$ defined by $f(t) = e^{it}$, $0 \le t \le 2\pi$. Show that the function f is one-to-one, onto, continuous but not a homeomorphism. (6.5)

$$\begin{bmatrix} 3 & -1 & 5 \\ 6 & 4 & 0 \\ 10 & 8 & 6 \end{bmatrix}$$

(b) Convert the following game problem into a pair of linear programming problems for players A and B and provide the optimal strategies of the two players and value of the game.

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 2 & 3 & 0 & -2 \\ 1 & 2 & 3 & 2 \end{bmatrix} \end{array}$$

(c) Find the inverse of the matrix using Simplex Method:

$$\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1228 **I**

Unique Paper Code : 2353010008

Name of the Paper : Linear Programming and Applications (DSE)

Name of the Course : **B.Sc. (H) Mathematics**

Semester : V

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.

1. (a) Solve the following linear programming problem by Graphical Method:

$$\begin{aligned} \text{Minimize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_2 \leq 4 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

(b) Examine the convexity of the set $S = \{(x_1, x_2): x_2^2 \geq 4x_1\}$. Justify your conclusion. Further prove or disprove that intersection of two convex sets is a convex set.

(c) Solve the following problem by simplex method:

$$\begin{aligned} \text{Minimize} \quad & x_1 + 2x_2 - 4x_3 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 \leq 9 \\ & x_1 + x_2 - x_3 \leq 2 \\ & -x_1 + x_2 + x_3 \leq 4 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

2. (a) Consider the following problem:

$$\begin{aligned} \text{Minimize} \quad & z = cx \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0, \end{aligned}$$

Let $(x_B, 0)$ be a basic feasible solution corresponding to basis B such that $z_j - c_j \leq 0$ for all non- basic variables x_j. Prove that $(x_B, 0)$ is an optimal basic feasible solution.

	I	II	III	IV	Availability
A	19	30	50	10	7
B	70	30	40	60	9
C	40	8	70	20	18
Requirements	5	8	7	14	

(c) Find the optimal solution of the Assignment Problem with the following cost matrix :

	U	V	X	Y	Z
I	2	9	2	7	1
II	6	8	7	6	1
III	4	6	5	3	1
IV	4	2	7	3	1
V	5	3	9	5	1

6. (a) (i) Define Saddle Point and Mixed Strategy for a "Two-Person Zero Sum" game.

(ii) Use maxmin and minmax Principle to find the saddle point, if it exists, for the following pay-off matrix

$$\text{Minimize } z = 2x_1 + 9x_2 + x_3$$
$$\text{subject to } \quad x_1 + 4x_2 + 2x_3 \geq 5$$
$$3x_1 + x_2 + 2x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0.$$

5. (a) For the following cost minimization Transportation Problem, find the initial basic feasible solution by using North West Corner Rule, Least Cost Method and Vogel's Approximation Method. Compare the three solutions in terms of cost:

Machine	I	II	III	IV	V	Supply
A	2	11	10	3	7	4
B	1	4	7	2	1	8
C	3	9	4	8	12	9
Demand	3	3	4	5	6	

(b) Solve the following cost minimization Transportation Problem:

(b) Find all the basic feasible solutions for the following system of equations:

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5.$$

Classify the obtained solutions as degenerate or non-degenerate.

(c) Reduce the feasible solution, $x_1 = 2$, $x_2 = 3$, $x_3 = 1$ of the following system to two different basic feasible solutions:

$$2x_1 + x_2 + 4x_3 = 11$$
$$3x_1 + x_2 + 5x_3 = 14.$$

3. (a) Solve the following problem by Big-M method

$$\text{Maximize } z = 6x_1 + 4x_2$$
$$\text{subject to } \quad 2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Is the solution unique? If not, give two different solutions.

(b) Solve the following problem by Two-phase method

$$\begin{aligned} &\text{Minimize } z = -5x_1 + 4x_2 - 3x_3 \\ &\text{subject to } \quad 2x_1 + x_2 - 6x_3 = 20 \\ &6x_1 + 5x_2 + 10x_3 \leq 76 \\ &8x_1 - 3x_2 + 6x_3 \leq 50 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

(c) Solve the following system of simultaneous equations using simplex method:

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2\,x_1 + x_2 &= 3 \end{aligned}$$

4 (a) (i) Prove that if either the primal or the dual problem has an unbounded objective function value, then the other problem has no feasible solution.

(ii) Show that the dual of the dual is primal for the following primal problem

$$\begin{aligned} &\text{Minimize } z = x_1 + 2x_2 - 3x_3 \\ &\text{subject to } \quad 4x_1 + 5x_2 - 6x_3 = 7 \\ &8x_1 - 9x_2 + 10x_3 \leq 11 \\ &x_1, x_2 \geq 0 \;, x_3 \text{ unrestricted.} \end{aligned}$$

(b) Obtain the dual of the following primal problem:

$$\begin{aligned} &\text{Minimize } z = 3x_1 - 2x_2 + 4x_3 \\ &\text{subject to } \quad 3x_1 + 5x_2 + 4x_3 \geq 7 \\ &6x_1 + x_2 + 3x_3 \geq 4 \\ &7x_1 - 2x_2 - x_3 \leq 10 \\ &x_1 - 2x_2 + 5x_3 \geq 3 \\ &x_1, x_2, x_3 \geq 0. \end{aligned}$$

(c) Solve the following problem using Complementary Slackness theorem

This question paper contains 6 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **1229**

Unique Paper Code : **2353010009**

Name of the Paper : **Mathematical Statstics**

Type of the Paper : **DSE**

Name of the Course : **B.Sc. (H) Mathematics**

Semester : **V**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Attempt any *two* parts from each question.

All questions carry equal marks.

Use of non-programmable scientific calculators and statistical tables is permitted.

Part A

1. (*a*) A large but sparsely populated country has two small hospitals, one at the south end of the country and the other at the north end. The south hospital's emergency room has 4 beds, whereas the north hospital's emergency room has only 3 beds. Let X denote the number of south beds occupied at a particular time on a given day, and let Y denote the number of north beds occupied at the same time on the same day. Suppose that these two rvs are independent, that the pmf of X puts probability masses. .1, .2, .3, *.2 and .2 on the x* values 0, 1, 2, 3, and 4, respectively, and that pmf of Y distributes probabilities .1, .3, .4, and .2 on the y values 0, 1, 2, and 3, respectively.

P.T.O.

(i) Display the joint pmf of X and Y in a joint probability table.

(ii) Compute $P(X \leq 1 \text{ and } Y \leq 1)$ and verify that this equals the product of $P(X \leq 1)$ and $P(Y \leq 1)$.

(iii) Express the event that the total number of beds occupied at the two hospitals combined is at most 1 in terms of X and Y, and then calculate this probability.

(iv) What is the probability that at least one of the two hospitals has no beds occupied ?

(b) An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X = the number of points earned on the first part and Y = the number of points earned on the second part. Suppose that the joint pmf of X and Y is given in the accompanying table.

		y			
	$p(x, y)$	**0**	**5**	**10**	**15**
	0	.02	.06	.02	.10
x	**5**	.04	.15	.20	.10
	10	.01	.15	.14	.01

(i) Compute the covariance for X and Y.

(ii) Compute ρ for X and Y.

(c) A concert has three pieces of music to be played before intermission. The time taken to play each piece has a normal distribution. Assume that the three times are independent of each other. The mean times are 15, 30, and 20 min, respectively, and the standard deviations are 1, 2, and 1.5 min, respectively. What is the probability that this part of the concert takes at most one hour ? Are there reasons to question the independence assumption ? Explain.

2. (*a*) A system consisting of two components will continue to operate only as long as both components function. Suppose the joint pdf of the lifetimes (months) of the two components in a system is given by $f(x, y) = c\,[10 - (x + y)]$, for $x > 0$, $y > 0$, $x + y < 10$. Find the value of the constant c. If the first component functions for exactly 3 months, what is the probability that the second functions for more than 2 months.

(*b*) Define the bivariate normal distribution. For a few years, the SAT consisted of three components : writing, critical reading, and mathematics. Let W = SAT Writing score and X = SAT Critical Reading score for a randomly selected student. According to the College Board, in 2012 W had mean 488 and standard deviation 114, while X had mean 496 and standard deviation 114. Suppose X and W have a bivariate normal distribution with Corr(X, W) = .5. If an English department plans to use X + W, a student's total score on the nonmath sections of the SAT to help determine admission. Then determine the distribution of X + W.

(*c*) (*i*) Suppose that the lifetimes of two components are independent of each other and that the first lifetime, X_1, has an exponential distribtuion with parameter $\lambda_1 = 1/1000$ whereas the second, X_2, has an exponential distribution with parameter $\lambda_2 = 1/1200$. Compute the probability that the sum of their lifetimes is at most 3000 h.

(*ii*) A surveyor wishes to lay out a square region with each side having length L. However, because of measurement error, he instead lays out a rectangle in which the north-south sides both have length X and the east-west sides both have length Y. Suppose that X and Y are independent and that each one is uniformly distributed on the interval [L – A, L + A] (where 0 < A < L). What is the expected area of the resulting rectangle ?

3. (a) A particular brand of dishwasher soap is sold in three sizes : 25, 40, and 65 oz. 20% of all purchasers select a 25-oz box, 50% select a 40-oz box, and the remaining 30% choose a 65-oz box. Let X_1 and X_2 denote the package sizes selected by two independently selected purchasers. Determine the sampling distribution of $\overline{X}$, calculate $E(\overline{X})$, and compare to μ.

(b) State the Central Limit Theorem.

When a batch of a certain chemical product is prepared, the amount of a particular impurity in the batch is a random variable with mean value 4.0 g and standard deviation 1.5 g. If 50 batches are indepedently prepared, what is the (approximate) probability that the sample average amount of impurity X is between 3.5 and 3.8 g ? Justify why is the Central Limit Theorem applicable here ?

(c) (i) Define chi-squared distribution with ν degrees of freedom.

(ii) Show that if $X_1, X_2, \ldots\ldots\ldots, X_n$ is a random sample from a $N(\mu, \sigma)$ distribution, then $(n - 1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Further deduce that $E(S^2) = \sigma^2$ and $V(S^2) = 2\sigma^4/(n-1)$.

4. (a) (i) Show that the sample variance S^2 is an unbiased estimator of σ^2 for any population distribution.

(ii) Define minimum variance unbiased estimator (MVUE).

(b) (i) The traditional criteria for "strong" passwords and the emerging advice is to use longer passphrases by concatenating several everyday words. Suppose that 10 students at a certain university are randomly selected, and it is found that the three of them use passphrases for their email accounts. Let p = P(passphrase); i.e., p is the proportion of all students at the university using a passphrase on their email accounts. Find the maximum likelihood estimate of the parameter p.

(ii) Suppose X has the pdf $f(x; \theta) = \theta x^{\theta-1}$ for $0 \leq x \leq 1$. Obtain the Fisher information $I(\theta)$.

(*c*) State the Neyman factorization theorem. Use it to show that the sufficient statistic for μ is $T = \Sigma X_i$, by considering a random sample $X_1, X_2, \ldots\ldots, X_n$ from a Poissson distribution with parameter μ, describing the numbers of errors in *n* batches of tax returns where each batch consists of many returns.

5. (*a*) Assume that the helium porosity (in percentage) of coal samples taken from any of coal samples taken from any particular seam is normally distributed with true standard deviation 75.

(*i*) Compute a 95% confidence interval (CI) for the true average porosity of certain seam if the average porosity for 20 specimens from the seam was 4.85.

[$Z_{0.05} = 1.96$]

(*ii*) Compute a 98% confidence interval (CI) for true average porosity of another seam based on 16 specimens with a sample average porosity of 4.56.

[$Z_{0.01} = 2.33$]

(*b*) For healthy individuals the level of prothrombin in the blood is approximately normal distributed with mean 20 mg/dL and standard deviation 4 mg/dL. Low levels indicate low clotting ability. In studying the effect of gallstones on prothrombin, the level of each patient in a sample is measured to see if there is a deficiency. Let μ be the true average level of prothrombin for gallstone partients (and assume $\sigma = 4$).

(*i*) What are the appropriate null and alternative hypotheses ?

(*ii*) Let $\overline{X}$ denote the sample average level of prothrombin in a sample of $n = 20$ randomly selected gallstone patients. Consider the test procedure with test statistic $\overline{X}$ and rejection region $\overline{x} \leq 17.92$. What is the probability distribution of the test statistic when H_0 is true ? What is the probability of a type I error for the test procedure ?

(c) On the label, Pepperidge Farm bagels are said to weigh four ounces each (113 g). A random sample of six bagels resulted in the following weights (in grams) :

117.6, 109.5, 111.6, 109.2, 119.1, 110.8.

Based on this sample, is there any reason to doubt that the population mean is at least 113 g at 0.05 level of significance ? [$t_{0.05,5}$ = 2.015]

6. (a) A plan for an executive traveler's club has been developed by an airline on the premise that 5% of its current customers would qualify for membership. A random sample of 500 customers yielded 40 who would qualify. Using this data, test at level 0.01 the null hypothesis that the company's premise is correct against the alternative that it is not correct. [$z_{0.01}$ = 2.576]

(b) The accompanying data on x = current density (mA/cm^2) and y = rate of deposition (mm/min) appeared in the article "Plating of 60/40 Tin/Lead Solder for Head Termination Metallurgy". Do you agree with the claim by the article's author that a linear relationship was obtained from the tin-lead rate of deposition as a function of current density ? Explain your reasoning.

X	20	40	60	80
Y	0.24	1.20	1.71	2.22

(c) A statistics department at a large university maintains a tutoring center for students in its introductory service courses. The center has been staffed with the expectation that 40% of its clients would be from the business statistics course, 30% from engineering statistics, 20% from the statistics course for social science students, and the other 10% from the course for agriculture students. A random sample of n = 120 clients revealed 52, 38, 21, and 9 from the four courses. Does this data suggest that the percentages on which staffing was based are not correct ? State and test the relevant hypotheses using α = 0.05. [$\chi^2_{0.05,3}$ = 7.815] at the tool crib. At what rate should the arrival of machinists to the tool crib increase to justify the addition of a second crib?

(8,7)

[This question paper contains 12 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5836 **I**

Unique Paper Code : 32371501

Name of the Paper : Stochastic Processes and Queuing Theory

Name of the Course : **B.Sc.(Hons.)**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.
2. Attempt **FIVE** questions in all
3. **Section I** is compulsory.
4. Attempt **FOUR** more questions, selecting **TWO** from each of the sections **II** and **III.**
5. Use of simple calculator is allowed.

Section I

1. (a) Let X be a discrete random variable with Probability Generating Function $P_X(s) = \frac{s}{5}(2 + 3s^2)$. Ascertain that $P_x(s)$ is a PGF. Find the distribution of X.

(b) Let e_1, e_2, ... be a sequence of independent, identically distributed random variables each with zero mean and variance σ^2. The observed time series, $\{Y_t: t = 1,2,...\}$, is constructed as follows:

$$Y_t = Y_{t-1} + e_t;$$

With initial condition $Y_1 = e_1$.. Find the auto covariance and autocorrelation function of Y_t.

customers in the system and average queue length.

(b) Arrivals of machinists at a tool crib are considered to be Poisson distributed at an average rate 6 per hour. The length of time the machinists must remain at the tool crib is exponentially distributed with average time of 0.05 hours.

(i) What is the probability that a machinist arriving at the tool crib will have to wait?

(ii) What is the average number of machinists at the tool crib?

(iii) The company will install a second tool crib when convinced that a machinist would have to spend 6 minutes in waiting and being served

(b) Divide the interval [0, t] into n (a large number) intervals of equal length h and a suppose that in each small interval, Bernoulli trials with probability of success λh and probability of failure (1-λh) are held. Show that the number of successes in an interval of length λt is a Poison process with mean At. State the assumptions you make. (8,7)

7. (a) A bank has a single teller (server) and a waiting area that can accommodate up N customers, including the one being served. The arrival of customers follows a Poisson process with rate λ (customers per minute). The service times are exponentially distributed with rate μ (services per minute). Derive the steady-state probabilities Pn for the number of customers in the system, where n = 0,1,2,..., N. Also, calculate average number of

(c) Find the PGF of geometric distribution.

(d) Show that the sum of two independent Poisson variates is also a Poisson variate.

(e) Let X be a non-negative integral valued random variable with probability $P(X=n)=p_n, n = 0,1,2,\ldots$ and probability generating function $P(s)=\sum_{k=0}^{\infty} p_k s^k$. Find the generating function for

(i) $P(X < n)$

(ii) $P(X > n + 1)$.

(f) At an airport security checkpoint, passengers arrive at a rate of λ =20 passengers per hour, and the time it takes to screen each passenger is exponentially distributed with a service rate of $\mu = 600$ passengers every day. Calculate the average number of passengers in the system. Also, what is the probability that exactly 5 passengers are in the system? $(5 \times 3 = 15)$

Section II

2. (a) A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4.

(i) Find the transition matrix for the chain.

(ii) The probability of his winning.

(b) Write the differential-difference equations for the Yule-Furry process. Obtain the expression for the size distribution $P_n(t) = P[N(t) = n]$, assuming that the process starts with one member at time $t = 0$. Further, obtain the PGF of $\{P_n(t)\}$.

(8,7)

6. (a) Define a Poisson process. State the postulates under which a count process will be $N_1(t)$ and $N_2(t)$ If are two independent series of Poisson a Poisson process. events, obtain the probability generating functions of

(i) $N_1(t)+N_2(t)$;

(ii) $N_1(t)-N_2(t)$.

(iii) Are the resultant processes Poisson again?

Find:

(i) $p_n = P\{X_n = 1\}$

(ii) $E\{X_n\}$ and $Var\{X_n\}$

(iii) $Corr\{X_{n-1}, X_n\}$.

(b) Define a persistent state and a transient state. Show that the state j is persistent iff

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty. \qquad (8,7)$$

Section III

5. (a) Define a classical ruin problem and obtain the following probabilities:

(i) The probability of gambler's ultimate ruin;

(ii) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

(b) Let $S_N = X_1 + X_2 + ... + X_N$, where N has Poisson distribution with mean a. If $X_i's$ i.i.d. Bernoulli distribution with Xis have $P[X_i = 1] = p$ and $P[X_i = 0] = 1 - p = q$, show that

(i) the conditional distribution of N given $S_N = y$ has probability mass function

$$P[N = n|S_N = y] = \frac{e^{-aq}(aq)^{n-y}}{(n-y)!},\ n \geq y.$$

(ii) Further $cov(N, S_N) = ap$. (8,7)

3. (a) Six boys Dev (D), Hemant (H), Jeet (J), M (Mohan), S (Sunil) and T (Tejas) play catch. If D has the ball, he is equally likely to throw it to H,

M, S and T. If H gets the ball, he is equally likely to throw it to D, J, S, T. Is S has the ball, he is equally likely to throw it to D, H, M and T. If either J or T gets the ball, they keep throwing it to each other. If M gets the ball, he runs away with it. Obtain the transition probability matrix

(b) Consider a Markov chain with three states, S={0, 1, 2} having the following transition probability matrix:

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

If the initial distribution $P\{X_0 = i\} = \frac{1}{3}$, i = 0, 1, 2.

Find

(i) $P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$

(ii) $P\{X_1 = 1\}$.

(iii) $P\{X_2 = 1, X_0 = 0\}$ (8, 7)

4. (a) Consider a Markov chain with state space {0,1}, having transition probability matrix:

$$P = \begin{bmatrix} 1-(1-c)p & (1-c)p \\ (1-c)(1-p) & (1-c)p+c \end{bmatrix}, \quad 0 < p < 1, 0 \le c \le 1,$$

and initial distribution $p1 = P\{X_0 = 1\} = p = 1 - P\{X_0 = 0\}$.