Indeterminate Forms & L Hopital's Rule : [redacted]

$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ in which $f(x) \to 0$ and $g(x) \to 0$ is called an indeterminate form of type $\frac{0}{0}$.

Similarly, the limit of a ratio, $\dfrac{f(x)}{g(x)}$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an indeterminate form of type $\frac{\infty}{\infty}$.

L' Hopital's rule for form $\frac{0}{0}$:

Suppose that f & g are differentiable fns. on an open interval containing $x = a$, except possibly at $x = a$ and that $\lim\limits_{x \to a} f(x) = 0$, $\lim\limits_{x \to a} g(x) = 0$.

If $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists or if this limit is $+\infty$ or $-\infty$,

then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$

Moreover this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, $x \to +\infty$.

L' Hopital's rule for form $\frac{\infty}{\infty}$:

Suppose that f and g are differentiable fns. on an open interval containing $x = a$, except possibly at $x = a$ and that $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$.

If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, $x \to +\infty$.

More Indeterminate forms :

There are few more indeterminate forms which are $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞.

Indeterminate form $0 \times \infty$ can sometimes be evaluated by rewriting the product as a ratio and then applying L' Hopital's rule for indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Similarly, indeterminate form of the type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate forms of types 0^0, ∞^0, 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = [f(x)]^{g(x)}$$

and then computing the limit of $\ln y$.

<u>Example</u>. Find $\lim_{x \to 0} (1 + \sin x)^{\frac{1}{x}}$

<u>Solution</u>: Let $y = (1 + \sin x)^{\frac{1}{x}}$

$$\Rightarrow \ln y = \frac{1}{x} \ln(1 + \sin x) = \frac{\ln(1 + \sin x)}{x}$$

$$\Rightarrow \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + \sin x)}{x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{\cos x}{1 + \sin x}}{1} \quad \text{(Applying L'Hospital's Rule)}$$

$$\lim_{x \to 0} \ln y = 1$$

$$\lim_{x \to 0} y = e^1$$

$$\text{or} \quad \boxed{\lim_{x \to 0} (1 + \sin x)^{\frac{1}{x}} = e}$$