Calculus of vector-valued functions:

Limit of a vector-valued functions:

Let $\vec{r}(t)$ be a vector-valued functions defined for all t in some open interval containg the number a, except that $\vec{r}(t)$ need not be defined at a. Then limit of $\vec{r}(t)$ at point a is written as

$$\lim_{x \to a} \vec{r}(t) = \vec{L}$$ if given any number $\varepsilon > 0$

we can find a number $\delta > 0$ such that

$$\| \vec{r}(t) - \vec{L} \| < \varepsilon \quad \text{if} \quad 0 < |t-a| < \delta .$$

Geometrical interpretation of limits:

If $\vec{r}(t)$ is a vector-valued fun^n then $\lim_{t \to a} \vec{r}(t) = \vec{L}$

if and only if the radius vector $\vec{r} = \vec{r}(t)$

approaches $\vec{L}$ in both length and direction as $t \to a$.

Continuity of a vector-valued function:

A vector-valued fun^n $\vec{r}(t)$ is continuous at $t = c$ provided:

(1.) $\vec{r}(c)$ is defined

(2.) $\lim_{t \to c} \vec{r}(t)$ exists

(3.) $\lim_{t \to c} \vec{r}(t) = \vec{r}(c)$

Question. Determine whether $\vec{r}(t)$ is continuous at $t=0$ where

(i) $\vec{r}(t) = (3\sin t)\,\hat{i} - 2t\,\hat{j}$

(ii) $\vec{r}(t) = t^2\,\hat{i} + \frac{1}{t}\,\hat{j} + t\,\hat{k}$

Solution. (i) Given $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$

$$= (3\sin t)\,\hat{i} + (-2t)\,\hat{j}$$

At $t=0$, $\vec{r}(0) = \vec{0}$

$$\lim_{t\to 0} \vec{r}(t) = \lim_{t\to 0}\left[(3\sin t)\,\hat{i} + (-2t)\,\hat{j}\right]$$

$$= \lim_{t\to 0}(3\sin t)\,\hat{i} + \lim_{t\to 0}(-2t)\,\hat{j}$$

$$= 0\cdot\hat{i} - 0\,\hat{j} = \vec{0} = \vec{r}(0)$$

Here all the three conditions of a function of being continuitious are satisfied. Therefore given funⁿ. is cts.

(ii) Given $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

$$= t^2\,\hat{i} + \frac{1}{t}\,\hat{j} + t\,\hat{k}$$

Here $\vec{r}(0)$ is not defined, as $y(t)$ is not defined at $t=0$. Therefore given funⁿ. is not continuous at $t=0$.

Also $\lim_{t\to 0}\vec{r}(t)$ does not exists as $\lim_{t\to 0} y(t)$ does not exist. □

Rules of Differentiation:

(i) $\dfrac{d}{dt}[\vec{c}] = 0$

(ii) $\dfrac{d}{dt}[k\,\vec{r}(t)] = k\left[\dfrac{d}{dt}\vec{r}(t)\right]$

(iii) $\dfrac{d}{dt}\left[f(t)\,\vec{r}(t)\right] = f(t)\dfrac{d}{dt}[\vec{r}(t)] + \dfrac{d}{dt}[f(t)]\,\vec{r}(t)$

(iv) $\dfrac{d}{dt}\left[\vec{r}_1(t)\cdot\vec{r}_2(t)\right] = \vec{r}_1(t)\cdot\dfrac{d\vec{r}_2(t)}{dt} + \dfrac{d\vec{r}_1(t)}{dt}\cdot\vec{r}_2(t)$

(v) $\dfrac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right] = \vec{r}_1(t)\times\dfrac{d\vec{r}_2(t)}{dt} + \dfrac{d\vec{r}_1}{dt}\times\vec{r}_2(t)$

Theorem. If $\vec{r}(t)$ is a vector-valued function in 2-space or 3-space and $\|\vec{r}(t)\|$ is constant for all t, then

$$\vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

Proof: Since $\dfrac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = \vec{r}(t)\cdot\dfrac{d\vec{r}}{dt} + \dfrac{d\vec{r}}{dt}\cdot\vec{r}$

or $\dfrac{d}{dt}\left[\|\vec{r}(t)\|^2\right] = 2\,\vec{r}(t)\cdot\dfrac{d\vec{r}}{dt}$

Since $\|\vec{r}(t)\|^2$ is constant, therefore $\dfrac{d}{dt}\left[\|\vec{r}(t)\|^2\right] = 0$

Hence $\vec{r}(t)\cdot\dfrac{d\vec{r}}{dt} = 0$

$\square$

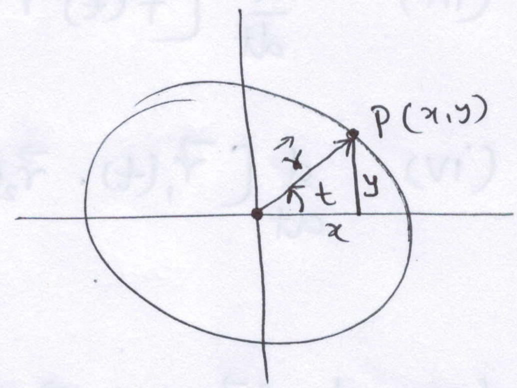
Question. Sketch the circle $\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j}$

and draw the vector with its correct length

(i) $\vec{r}\,'\left(\frac{\pi}{4}\right)$ (ii) $\vec{r}\,''(\pi)$ (iii) $\vec{r}(2\pi) - \vec{r}\left(\frac{3\pi}{2}\right)$

Sol^n. (i) Given $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$

$$= (\cos t)\,\hat{i} + (\sin t)\,\hat{j}$$



Here $x = \cos t$, $y = \sin t$

$$x^2 + y^2 = 1$$

which is a circle with centre $(0,0)$ and radius 1.

Now $\vec{r}\,'(t) = \left\{\frac{d}{dt}(x(t))\right\}\hat{i} + \left\{\frac{d}{dt}(y(t))\right\}\hat{j}$

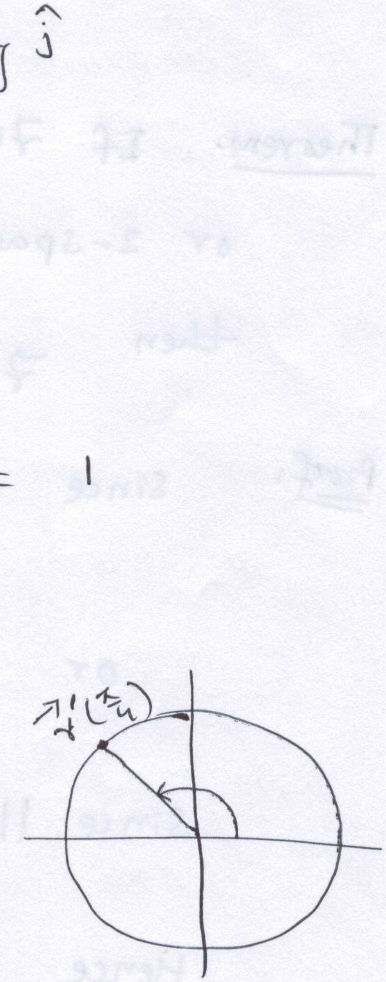
$$\vec{r}\,'(t) = (-\sin t)\,\hat{i} + (\cos t)\,\hat{j}$$

$$\vec{r}\,'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\,\hat{i} + \frac{1}{\sqrt{2}}\,\hat{j}$$

$$\left\|\vec{r}\,'\left(\frac{\pi}{4}\right)\right\| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

Here $x = r\cos\theta = -\frac{1}{\sqrt{2}}$

$$y = r\sin\theta = \frac{1}{\sqrt{2}}$$



$$\therefore \tan\theta = \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$$

$$\therefore \theta = \tan^{-1}(-1) = \tan^{-1}\tan\left(\pi - \frac{\pi}{4}\right)$$

$$\theta = \frac{3\pi}{4}$$