Example: $S = \{x : 0 < x < 1\}$.

upper bounds are $1, 1.5, 2, 3, \ldots$

(b) The set S is called bounded below if there exists a real number $w \in S$ such that $w \leq s \ \forall s \in S$.

This number w is called lower bound of S.

Example: $S = \{x : 0 < x < 1\}$

Lower bounds are $0, -0.1, -0.2, -1, -2, \ldots$

(c) The set S is called bounded if it is both bounded above and bounded below.

Example: $S = \{x : 0 < x < 1\}$ is a bounded set.

(d) The set S is called unbounded if it is not bounded.

Example: (i) $S = \{x : x \geq \frac{1}{2}\}$ is unbounded, because it is not bounded above.

(ii) $S = \{x : x \leq 3\}$ is unbounded, because it is not bounded below.

Supremum (Least upper bound):

Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then a number u is called Supremum of S if it satisfies the following Cond's:

(i) u is the upper bound of S

(ii) If v is any upper bound of S, then $u \leq v$.

Notation: Sup. $S = u$.

Example: let $S = \{x : 0 < x < 1\}$.

Sup. $S = 1$.

upper bounds are 1, 1.5, 2, 3, ---

least upper bound is 1.

Infimum (Greatest lower bound):

Let S be a nonempty subset of $\mathbb{R}$. If S is bounded below, then a number w is called infimum of S if it satisfies the following Conditions.

(i) w is the lower bound of S

(ii) if v is any other lower bound of S, then $v \leq w$.

Notation: Inf. $S = w$

Example: $S = \{x : 0 < x < 1\}$

Inf. $S = 0$

Lower bounds are.

$\boxed{0}$, -0.1, -0.2, ---, -1, -2, ---

Note:

① There can be only one supremum and only one infimum for any set S.

② sup. S is unique, if it exists.

③ inf. S is unique, if it exists

④ sup. S and inf. S may or may not belong to S.

The completeness property of $\mathbb{R}$:

Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

This property is also called supremum property of $\mathbb{R}$.

Note: Every nonempty set of real numbers that has a lower bound also has an infimum in $\mathbb{R}$.

Q. Determine the supremum and infimum for the following sets:

(i) $S_1 = \{x : x \geq \frac{1}{2}\}$ → sup. S_1 does not exist → inf. $S_1 = 1/2$

(ii) $S_2 = \{x : \frac{1}{2} \leq x \leq 2\}$ → sup. $S_2 = 2$ → inf. $S_2 = 1/2$

(iii) $S_3 = \{\frac{1}{n} : n \in \mathbb{N}\}$ → sup. $S_3 = 1$ → inf. $S_3 = 0$

(iv) $S_4 = \left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$ $\Rightarrow$ sup. $S_4 \to 2$

$\Rightarrow$ inf. $S_4 \to 1$

(v) $S_5 = \{ x \in \mathbb{R} : x > 0 \}$ $\Rightarrow$ sup. S_5 does not exist

$\Rightarrow$ inf. $S_5 = 0$

(vi) $S_6 = \{ x \in \mathbb{R} : |x-2| \leq x+1 \}$.

$= \{ x \in \mathbb{R} : x \geq \tfrac{1}{2} \} = S_1$

$S_3 = \{ \frac{1}{n} : n \in \mathbb{N} \}$

$= \{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \}$

$=$

infimum = greatest lower bound.