11.8

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Example-2 $0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$

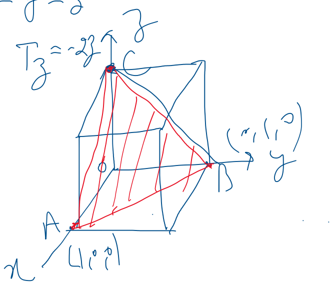
$x+y+z=1$ ✓✓

$T(x,y,z)=4-2x^2-y^2-z^2$ ✓✓

$T_x=-4x,\ T_y=-2y,\ T_z=-2z$

$g(x,y,z)=x+y+z$

$g_x=1=g_y=g_z$



$T_x=\lambda g_x,\ T_y=\lambda g_y,\ T_z=\lambda g_z,\ g(x,y,z)=1$

$-4x=\lambda,\ -2y=\lambda,\ -2z=\lambda,\ x+y+z=1$

$\Rightarrow x=-\frac{\lambda}{4},\ y=-\frac{\lambda}{2},\ z=-\frac{\lambda}{2}$

$-\frac{\lambda}{4}-\frac{\lambda}{2}-\frac{\lambda}{2}=1 \Rightarrow -\frac{\lambda}{4}-\lambda=-\frac{5}{4}\lambda=1$

$\lambda=-\frac{4}{5}$

$x=\frac{1}{5},\ y=\frac{2}{5},\ z=\frac{2}{5}$ $\left(\frac{1}{5},\frac{2}{5},\frac{2}{5}\right)$

Edge AB $x+y=1,\ z=0,\ T_1(x)=4-2x^2-y^2$

$=4-2x^2-(1-x)^2$

$=4-2x^2-(1+x^2-2x)=3-3x^2+2x$

$\therefore T_1(x)=3-3x^2+2x,\quad 0\le x\le 1$

BC, $y+z=1$, $x=0$, $T_2(y) = 4-y^2-(1-y)^2$, $0\le y\le 1$

$3-2y^2+2y$

AC, $x+z=1$, $y=0$, $T_3(x) = 4-2x^2-(1-x)^2$, $0\le x\le 1$

$3-3x^2+2x$

$T_1'(x) = 0 \Rightarrow -6x+2=0 \Rightarrow x = +\frac{1}{3}$

$y = 1-x = \frac{2}{3}$ $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$

$T_2'(y) = 0 \Rightarrow -4y+2 \Rightarrow y = \frac{1}{2}$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$

$T_3'(x) = 0 \Rightarrow -6x+2=0 \Rightarrow x = \frac{1}{3}$

T $y=0$, $z = \frac{2}{3}$ $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$

$\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ — ✓ maximum

$\left(\frac{1}{3}, \frac{2}{3}, 0\right)$ —

$\left(0, \frac{1}{2}, \frac{1}{2}\right)$ —

$\left(\frac{1}{3}, 0, \frac{2}{3}\right)$ —

$(1, 0, 0)$ — ✓ minimum

$(0, 1, 0)$ —

$(0, 0, 1)$ —

$f(x,y)$, $g(x,y) = C$ ✓ ✓ | (x_0, y_0) is extremum if c is changed by a unit

(x_0, y_0) will change.

$E = f(x_0, y_0)$ will also change

$\frac{dE}{dc}$; change in extreme value of f

Corresponding to unit change in c

Let us find out ;

Theorem. Rate of change of extreme value:

Suppose E is extreme value (max or min) of f subject to constraint $g(x, y) = c$. Then the Lagrange multiplier λ is the rate of change of E with respect to c

$$\frac{dE}{dc} = \lambda$$

Proof : Let be (x, y) be extreme pt $\Rightarrow E = f(x, y)$

at extreme pt

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = c$$

Now $x = x(c)$, $y = y(c)$, $E = E(x, y)$.

i.e x, y depend on c and E depends on x, y.

$$\frac{dE}{dc} = \frac{\partial E}{\partial x} \cdot \frac{dx}{dc} + \frac{\partial E}{\partial y} \frac{dy}{dc} \quad (\text{by chain rule})$$

$E = f(x, y)$

$\frac{\partial E}{\partial x} = f_x$, $\frac{\partial E}{\partial y} = f_y$

$$= f_x \frac{dx}{dc} + f_y \frac{dy}{dc}$$

$$= \lambda g_x \frac{dx}{dc} + \lambda g_y \frac{dy}{dc}$$

$$= \lambda \left(g_x \frac{dx}{dc} + g_y \frac{dy}{dc} \right) = \lambda \frac{dg}{dc}$$

$$= \lambda \frac{dg}{dc}$$

$$\left(g(x, y) = c \quad \frac{dg}{dc} = \frac{dc}{dc} = 1 \right.$$

$$= \lambda$$

$$\boxed{\frac{dE}{dc} = \lambda}$$

Example 3

x thousand \$ labour ✓

y 〃 \$ equipment ✓

$\underline{Q(x,y)} = 50x^{2/5}y^{3/5}$ unit

$\underline{150,000\ \$}$

$\underline{x+y=150}$

maximize $Q(x,y) = 50x^{2/5}y^{3/5}$, subject to constraint $x+y=150$

$$g(x,y) = x+y$$

$Q_x = \lambda g_x$, $Q_y = \lambda g_y$ ∋ $g(x,y) = c$

$50\times\frac{2}{5}x^{-3/5}y^{3/5} = \lambda$ — ①, $50\times\frac{3}{5}x^{2/5}y^{-2/5} = \lambda$ — ②

$$20x^{-3/5}y^{3/5} = \lambda = 30x^{2/5}y^{-2/5}$$

$$x+y=150 \text{ — ③}$$

$\Rightarrow$ $2y = 3x \Rightarrow y = 1.5x$

$x + 1.5x = 150 \Rightarrow 2.5x = 150$

$\Rightarrow x = 60$

$\Rightarrow y = 90$

60,000 — labour

90,000 — equipment then the output will be max.

$\therefore Q = 50(60)^{2/5}(90)^{3/5} \approx \underline{3826.27}$

$$\underline{150,000} \longrightarrow \underline{1,51,000}$$

C ~~is~~ increasing 1 unit

$$\lambda \Rightarrow 20\, x^{-3/5} \times y^{3/5}$$

$$= 20\,(60)^{-3/5}\,(90)^{3/5}$$

$$\approx \boxed{25.508} = \frac{dE}{dC}$$

$$\text{new } E = \underline{3826.27} + \underline{25.508}$$

$$=$$